$a = 2\alpha$, $b = 2\beta$, where α and β are integers. Then from (1) we have

$$(2) \qquad x = \alpha + \beta, \qquad y = \alpha - \beta, \qquad n = 2\alpha\beta.$$

If these values of x, y, n are substituted in the original equations we find that

$$(3) \qquad z = \alpha^2 + \beta^2.$$

From these computations one readily concludes as follows:

A necessary and sufficient condition on the positive integer n is that it shall be even. All non-negative solutions for a particular value of the even integer n may be found as follows: Separate $n/2$ in every way possible into the product of two positive integral factors α and β such that $\alpha \geqq \beta$. For each such separation the corresponding values of x, y, z are given by equations (2) and (3).

It is obvious that the solution is unique when and only when $n/2$ is a prime number.

Also solved by C. F. Gummer, Elijah Swift, Horace Olson, E. B. Escott, J. L. Riley, C. E. Githens and V. M. Spunar.

252. Proposed by E. J. Moulton, Northwestern University.

(*A*) Show that the number of integers x on the interval $10^r \leqq x < 10^{r+1}$ which do not contain the digit 1 at least p times, $p \leqq r$, is

$$9\cdot\{[\text{first } p \text{ terms of expansion of } (9+1)^r] - {}_rC_{p-1}\cdot 9^{r-p}\},$$

where ${}_rC_{p-1}$ is the coefficient of x^{p-1} in the expansion of $(1+x)^r$.

(*B*) Show that the number of integers x on the interval $10^r \leqq x < 10^{r+1}$ which do not contain the digit 0 at least p times, $p \leqq r$, is

$$9\cdot[\text{first } p \text{ terms of expansion of } (9+1)^r].$$

Solution by C. C. Yen, Tangshan, North China.

The proof of (*B*), being simpler, is given first.

(*B*) The total number of integers x on the interval $10^r \leqq x < 10^{r+1}$ is $(10^{r+1} - 10^r) = 9\cdot 10^r$. If the whole interval be divided into nine subintervals, $n\cdot 10^r \leqq x < (n+1)\cdot 10^r$, $n = 1, 2, \cdots$ up to 9; then each subinterval will contain 10^r or $(9+1)^r$ integers.

Consider each of the subintervals. First, there is one integer which contains the digit 0 r times. Next there are ${}_rC_{r-1}\cdot 9$ integers which contain the digit 0 $(r-1)$ times; for out of the r digit places $(r-1)$ of them may be filled by 0's in ${}_rC_{r-1}$ ways; and the remaining place may be filled by any one of the digits $1, 2, 3, \cdots, 9$, so that each of the ${}_rC_{r-1}$ ways gives rise to 9 such integers. In the same way, $(r-2)$ of the r places may be filled by 0's in ${}_rC_{r-2}$ different ways; and each of the two remaining places may be filled by any one of the digits $1, 2, \cdots 9$, so that each of the ${}_rC_{r-2}$ ways gives rise to 9^2 integers; hence, there are ${}_rC_{r-2}\cdot 9^2$ integers containing the digit 0 $(r-2)$ times, and so on. In general, p out of the r places may be filled by 0's in ${}_rC_p$ different ways, each of which gives rise to 9^{r-p} integers; the number of integers containing the digit 0 p times is, therefore, ${}_rC_p\cdot 9^{r-p}$. Hence, we have the result that in each subinterval there are

$$(1) \qquad {}_rC_p\cdot 9^{r-p} + \cdots + {}_rC_{r-3}\cdot 9^3 + {}_rC_{r-2}\cdot 9^2 + {}_rC_{\cdot-1}\cdot 9 + 1$$

integers containing the digit 0 at least p times, $p \leqq r$. But expression (1) consists of the last $r - p + 1$ terms of the expansion of $(9+1)^r$; and, as we have seen, the number of integers in each subinterval is $(9+1)^r$. It follows therefore that the number of integers which do not contain the digit 0 at least p times ($p \leqq r$) in each subinterval is given by the first p terms of the expansion $(9+1)^r$, and hence in the whole interval $10^r \leqq x < 10^{r+1}$, the number is

$$9\cdot[\text{first } p \text{ terms of expansion of } (9+1)^r],$$

as was to be proved.

(*A*) To prove (*A*), let the interval $10^r \leqq x < 10^{r+1}$ be divided into subintervals as before. Then in each of the subintervals, except the first, similar reasonings show that the number of integers containing the digit 1 at least p times, $p \leqq r$, is expressed by (1). In the first subinterval, $10^r \leqq x < 2\cdot 10^r$, the first digit of every integer x is 1, so that (1) gives the number of integers containing the digit 1 at least $(p+1)$ times. Now, out of the r digit places (beginning from the

second), $(p-1)$ may be filled by 1's in ${}_rC_{p-1}$ ways, and each of these different ways gives rise to $9^{r-(p-1)}$ integers containing the digit 1 p times. Hence, in the first subinterval the number of integers containing 1 at least p times is equal to the sum of expression (1) and ${}_rC_{p-1}\cdot 9^{r-p+1}$. And, therefore, the number of integers in the interval $10^r \leqq x < 10^{r+1}$ which do not contain the digit 1 at least p times, $p \leqq r$, is

$$9\cdot[\text{first } p \text{ terms of expansion of } (9+1)^r] - {}_rC_{p-1}\cdot 9^{r-p+1},$$

or

$$9\cdot\{[\text{first } p \text{ terms of expansion of } (9+1)^r] - {}_rC_{p-1}\cdot 9^{r-p}\}.$$

253. Proposed by HERBERT N. CARLETON, West Newbury, Massachusetts.

Prove that $n^{2k+8} - n^{2k} \equiv 0 \pmod{20}$ for integral values of n and k.

SOLUTION BY R. M. MATHEWS, Riverside, California.

$$n^{2k+8} - n^{2k} = n^{2k}(n^2-1)(n^2+1)(n^4+1)$$

When n is even, $n^{2k} \equiv 0 \pmod 4$. When n is odd, $n^2 - 1 \equiv 0 \pmod 4$.

Next, n being an integer must be of the form $5m$, $5m \pm 1$, or $5m \pm 2$.

For n of the form $5m$, $n^{2k} \equiv 0 \pmod 5$; for n of the form $5m \pm 1$, $n^2 - 1 \equiv 0 \pmod 5$; and for n of the form $5m \pm 2$, $n^2 + 1 \equiv 0 \pmod 5$.

Hence, $n^{2k+8} - n^{2k} \equiv 0 \pmod{20}$, n and k being integers. This is also true of $n^{2k+4} - n^{2k}$.

Also solved by O. S. ADAMS, W. J. THOME, ELIJAH SWIFT, E. B. ESCOTT, C. C. YEN, and the PROPOSER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

REPLIES.

34. Given the mixed integral and functional equation

$$\int_{x=0}^{x=x} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{x}{2}\right) + f(x)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

REMARK BY S. A. COREY, Albia, Iowa.

The prismoidal formula gives the exact value of this integral whenever the fourth derivative of $f(x) = 0$. This was shown in an article entitled "Certain Integration Formulæ Useful in Numerical Computation" in Vol. XIX, Nos. 6 and 7, of this MONTHLY, in which formula $(1r)$ is the prismoidal formula including an expression for the remainder term.

$f(x) = Ax^3 + Bx^2 + Cx + D$ is, therefore, the most general value of the function $f(x)$ for which the prismoidal formula gives the exact value for all values of x.

28. Is it possible to obtain $\int \cos\theta^2 d\theta$ without expanding $\cos\theta^2$? If it is not, can some interesting properties of this integral be determined by treating it as a special function?